Cavity losses for the dissipative Jaynes–Cummings Hamiltonian beyond rotating wave approximation

2007 J. Phys. A: Math. Theor. 40 14527

(http://iopscience.iop.org/1751-8121/40/48/015)

J. Phys. A: Math. Theor. **40** (2007) 14527–14536 doi:10.1088/1751-8113/40/48/015

Cavity losses for the dissipative Jaynes–Cummings Hamiltonian beyond rotating wave approximation

M Scala[1], B Militello[1], A Messina[1], S Maniscalco[2], J Piilo[2] and K-A Suominen[2]

[1] MIUR and Dipartimento di Scienze Fisiche ed Astronomiche dell'Università di Palermo, via Archirafi 36, I-90123 Palermo, Italy
[2] Department of Physics, University of Turku, FI-20014 Turun Yliopisto, Finland

E-mail: matteo.scala@fisica.unipa.it

Received 10 September 2007
Published 14 November 2007
Online at stacks.iop.org/JPhysA/40/14527

Abstract
A microscopic derivation of the master equation for the Jaynes–Cummings model with cavity losses is given, taking into account the terms in the dissipator which vary with frequencies of the order of the vacuum Rabi frequency. Our approach allows us to single out physical contexts wherein the usual phenomenological dissipator turns out to be fully justified and constitutes an extension of our previous analysis (Scala *et al* 2007 *Phys. Rev.* A **75** 013811), where a microscopic derivation was given in the framework of the rotating wave approximation.

PACS numbers: 42.50.Lc, 03.65.Yz, 42.50.Pq

1. Introduction

Since its first appearance [1], the Jaynes–Cummings (JC) model has been extensively used to study radiation–matter interaction [2] in contexts like cavity quantum electrodynamics (CQED) [3–5] and ion traps [6]. In a CQED scenario [3], for instance, the model describes the one-photon interaction between a two-level atom and a quantized normal mode of the electromagnetic cavity field, while in ion traps [6] it describes the interaction between a two-level atom and a vibrational mode of its center of mass. In both contexts, the quantum effects predicted by the model, such as Rabi oscillations and collapses and revivals of the atomic inversion operators, have been experimentally observed.

Concentrating on CQED, it is of central importance to describe cavity losses, due for example to dissipative effects in the cavity mirrors. In [7] we addressed this problem and provided a microscopic derivation for the master equation for the JC model neglecting spontaneous emission and supposing that the cavity was coupled to a bosonic reservoir. The derived master equation was then compared to the phenomenological master equation used

in the literature [3] and some physical examples were given to show that the two models describe very different decay mechanisms. Indeed, while in the phenomenological model it is the cavity only which directly decays and the atom loses energy through its coupling with the cavity mode, in our model it is the atom–cavity system as a whole which leaks, although only the cavity is coupled to an external environment.

Our approach in [7] relied on the rotating wave approximation (RWA) performed on the dissipator, as done in the usual formalism of master equations [8]. In fact it is possible to perform RWA in our case only when the vacuum Rabi frequency is much larger than the largest decay rate involved, i.e., in the strong coupling regime. Since the phenomenological model is claimed to be valid also in regimes wherein the vacuum Rabi frequency and the cavity decay rate are of the same order, the comparison between it and the microscopic model will be complete only when a microscopic derivation is given which takes into account in the dissipator all the terms oscillating at frequencies of the order of the vacuum Rabi frequency.

This is what we are going to do in this paper. Our present investigation, based on a deeper analysis of the terms neglected in the so-called dressed-state approximation performed on the phenomenological model [3], shows that the microscopic and the phenomenological model can be much closer than one might think in the context of our previous RWA model. In particular we will see that the two models may coincide if the spectrum of the environment is flat, even for small nonzero temperatures.

The paper is structured as follows. In section 2 we review the JC model, the usual phenomenological way to describe cavity losses and our microscopic model derived in [7], discussing its validity and limitations. In section 3 we derive the microscopic model beyond RWA and in section 4 the latter is compared to the phenomenological model, singling out the conditions under which the two models coincide, both exactly and approximately. Finally in section 5 some conclusive remarks are given.

2. The Jaynes–Cummings model: the phenomenological description of cavity losses

In [7] we addressed the problem of the description of the dissipation and decoherence processes in a two-level atom–cavity system, neglecting spontaneous emission, and taking into account the coupling between the cavity and an external bosonic environment. Denoting by $|g\rangle$ and $|e\rangle$ the atomic ground and excited states respectively, and calling ω_0 the atomic Bohr frequency, the atom–cavity interaction, in the RWA and in units of $\hbar$, at resonance, is described by the Jaynes–Cummings Hamiltonian [1]:

$$H_{\rm JC} = \frac{\omega_0}{2}\sigma_z + \omega_0 a^\dagger a + \Omega\,(a\sigma_+ + a^\dagger\sigma_-). \tag{1}$$

Here $a^\dagger$ (a) denotes the creation (annihilation) operator of the electromagnetic mode, and the atomic degrees of freedom are described by $\sigma_- = |g\rangle\langle e|$, $\sigma_+ = |e\rangle\langle g|$, and $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$. The eigenstates and eigenvalues of $H_{\rm JC}$ are [2]

$$|E_{N,\pm}\rangle = \tfrac{1}{\sqrt{2}}(|N, g\rangle \pm |N-1, e\rangle), \qquad E_{N,\pm} = \left(N - \tfrac{1}{2}\right)\omega_0 \pm \Omega\sqrt{N}, \tag{2}$$

except for the ground state which is given by

$$|E_0\rangle = |0, g\rangle, \qquad E_0 = -\frac{\omega_0}{2}, \tag{3}$$

where $|N, i\rangle = |N\rangle|i\rangle$, with $i = e, g$, indicates the tensor product of the Fock state $|N\rangle$, with $a^\dagger a|N\rangle = N|N\rangle$, and the electronic state $|i\rangle$.

In the usual treatment of the problem of cavity losses one derives microscopically the master equation for the cavity only, when it is coupled to an environment at temperature T

[9] and then assumes that the atom inside the cavity causes only a change in the Hamiltonian governing the unitary part of the dynamics [3]. In this way the following phenomenological master equation for the density operator of the atom–cavity system is assumed to be valid:

$$\dot{\rho} = -\mathrm{i}[H_{\mathrm{JC}}, \rho] + \gamma[n(\omega_0) + 1]\left[a\rho a^\dagger - \tfrac{1}{2}(a^\dagger a\rho + \rho a^\dagger a)\right] + \gamma n(\omega_0)\left[a^\dagger \rho a - \tfrac{1}{2}(aa^\dagger\rho + \rho aa^\dagger)\right], \tag{4}$$

where $n(\omega_0)$ is the average number of quanta of the reservoir in the quantized mode of frequency ω_0, and γ is the rate of loss of cavity photons. For a reservoir at zero temperature, equation (4) becomes

$$\dot{\rho} = -\mathrm{i}[H_{\mathrm{JC}}, \rho] + \gamma\left(a\rho a^\dagger - \tfrac{1}{2}a^\dagger a\rho - \tfrac{1}{2}\rho a^\dagger a\right). \tag{5}$$

Equations (4) and (5) are the phenomenological models that are very often used in the literature on cavity QED [4, 10–15].

In [7] we investigated the legitimacy of equation (4). By applying the general formalism from the theory of open quantum systems [8], we microscopically derived the master equation for the system under scrutiny assuming from the very beginning that the Hamiltonian of the system was the JC model.

Stated another way, the general theory of master equations in the Born–Markov approximation claims that the decay channels are described by jump operators which are transition operators between eigenstates of the Hamiltonian of the system, with coefficients depending on the details of the coupling between the system and the environment. This feature of the theory is quite general and holds even when the system is bipartite and only one of its two parts is coupled to an external environment.

By applying this formalism to the JC model, we found [7] that, although cavity only is coupled to an external environment, it is instead the atom–cavity system as a whole which decays, the irreversible transitions giving rise to dissipation and decoherence through jumps between dressed states. We then investigated the time evolution of the atom–cavity system predicted by the phenomenological model and by our microscopic model to clarify the different physical mechanisms underlying the two different models.

We stress that this approach includes the so-called rotating wave approximation (RWA)[3] on the dissipator of the master equation and is at the origin of its limited range of applicability expressible as [7]

$$2\Omega \gg \gamma_{\max}. \tag{6}$$

In other words the RWA in [7] is done over timescales of the order of the inverse of the Rabi frequency 2Ω of the atom–cavity system, and the microscopic master equation is valid only when the decay occurs on a timescale which is much longer than the Rabi period. On the other hand the phenomenological model described by equations (4) and (5) uses a dissipator on which the RWA has been done over frequencies which are of the order of ω_0, which is much larger than 2Ω: therefore the phenomenological model is claimed to be valid in a much larger range of values of the decay rate, since in this model one must have

$$\omega_0 \gg \gamma. \tag{7}$$

In the following we investigate how the master equation for the JC model is changed when its microscopic derivation does not neglect those terms in the dissipator oscillating at frequencies of the order of the Rabi frequency.

[3] One should avoid confusion between the RWA performed on the Hamiltonian in the JC model and the RWA performed on the dissipator. In the following we will always refer to RWA as to the operation of neglecting the oscillating terms in the dissipator in the interaction picture when deriving the master equation.

3. The microscopic master equation for the JC model beyond the RWA

In [7] we have assumed the validity of the RWA on timescales of the order of the period of the Rabi oscillations. Now we instead take into account the terms in the dissipator oscillating at frequencies proportional to Ω, in order to obtain a microscopic model suitable for a comparison with the phenomenological model in all damping regimes.

What we are going to do now is to skip the step corresponding to RWA in the general formalism to get a master equation under the Born and Markov approximations and to see what we can say about the master equation for the JC model starting from the following model for the total closed system:

$$H_S = H_{\rm JC}, \qquad H_E = \sum_k \omega_k b_k^\dagger b_k, \qquad H_{\rm int} = (a + a^\dagger) \sum_k g_k \left(b_k + b_k^\dagger\right), \tag{8}$$

which describes an atom–cavity system wherein the cavity is coupled to a bosonic environment through the interaction Hamiltonian $H_{\rm int}$. This model is equivalent to that used in [9], since the additional terms rapidly oscillating at frequencies of the order of ω_0 are washed out in the total and partial RWAs performed in the following.

From the interaction Hamiltonian $H_{\rm int}$ one gets the jump operators, according to the recipe given in [8]:

$$\begin{aligned} A(E_{N',l} - E_{N,m}) &= |E_{N,m}\rangle\langle E_{N,m}|(a + a^\dagger)|E_{N'l}\rangle\langle E_{N',l}| \\ &= \tfrac{1}{2}\delta_{N,N'-1}(\sqrt{N+1} + lm\sqrt{N})|E_{N,m}\rangle\langle E_{N+1,l}|, \end{aligned} \tag{9}$$

for $N \geqslant 1$ and

$$A(E_{1,\pm} - E_0) = \tfrac{1}{\sqrt{2}}|E_0\rangle\langle E_{1,\pm}|, \tag{10}$$

for $N = 0$. In equation (9) we indicate the states $|E_{N,\pm}\rangle$ by $|E_{N,\pm 1}\rangle$ and the energy eigenvalues $E_{N,\pm}$ by $E_{N,\pm 1}$. Accordingly l and m take the values ± 1. So all the possible jump operators are transition operators describing jumps between eigenstates (dressed states) of $H_{\rm JC}$ differing for one excitation only.

We will use the following notation for the jump operators:

$$A_{Nlm} = A(E_{N+1,l} - E_{N,m}) \tag{11}$$

for $N \geqslant 1$ and

$$A_{0l} = A(E_{1,l} - E_0) \tag{12}$$

for $N = 0$.

Before explicitly inserting these operators in the master equation, let us examine the structure of the equation for the reduced density operator of the atom–cavity system, after the Born and Markov approximations. The equation can be cast in this form:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = &\sum_{\omega,\omega' \geqslant 0} \{\mathrm{e}^{i(\omega' - \omega)t}\Gamma(\omega)[A(\omega)\rho(t)A^\dagger(\omega') - A^\dagger(\omega')A(\omega)\rho(t)] + \text{h.c.}\} \\ &+ \sum_{\omega,\omega' > 0} \{\mathrm{e}^{i(\omega - \omega')t}\Gamma(-\omega)[A^\dagger(\omega)\rho(t)A(\omega') - A(\omega')A^\dagger(\omega)\rho(t)] + \text{h.c.}\} \\ &+ \sum_{\omega \geqslant 0,\omega' > 0} \{\mathrm{e}^{-i(\omega' + \omega)t}\Gamma(\omega)[A(\omega)\rho(t)A(\omega') - A(\omega')A(\omega)\rho(t)] + \text{h.c.}\} \\ &+ \sum_{\omega > 0,\omega' \geqslant 0} \{\mathrm{e}^{i(\omega' + \omega)t}\Gamma(-\omega)[A^\dagger(\omega)\rho(t)A^\dagger(\omega') - A^\dagger(\omega')A^\dagger(\omega)\rho(t)] + \text{h.c.}\}, \end{aligned} \tag{13}$$

where we have explicitly written the minus sign for each negative Bohr frequency. The coefficients $\Gamma(\omega)$ are complex functions of the particular Bohr frequency involved and are equal to

$$\Gamma(\omega) = \int_0^{+\infty} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau} \langle E(\tau)E(0)\rangle, \tag{14}$$

with $E(\tau) = \sum_k g_k\big(b_k\, \mathrm{e}^{-\mathrm{i}\omega_k\tau} + b_k^\dagger\, \mathrm{e}^{\mathrm{i}\omega_k\tau}\big)$.

All the positive Bohr frequencies involved in the master equation can be written in the following way:

$$\omega_{Nlm} = E_{N+1,l} - E_{N,m} = \omega_0 + (l\sqrt{N+1} - m\sqrt{N})\Omega, \tag{15}$$

from which, remembering that $\omega_0 \gg \Omega$, it follows that the first two rows in the master equation are composed of terms which are either stationary or oscillating at frequencies proportional to the Rabi frequency, since

$$\omega_{Nlm} - \omega_{N',kn} = (l\sqrt{N+1} - m\sqrt{N} - k\sqrt{N'+1} + n\sqrt{N'})\Omega, \tag{16}$$

while the last two rows are composed of terms oscillating at much larger frequencies, since

$$\omega_{Nlm} + \omega_{N',kn} = 2\omega_0 + (l\sqrt{N+1} - m\sqrt{N} + k\sqrt{N'+1} - n\sqrt{N'})\Omega. \tag{17}$$

Now we will neglect only the last two rows in equation (13), i.e., all the rapidly oscillating terms, whereas in [7] we neglected all the time-dependent terms, including the slow ones: we will call *quasi-RWA master equation* the resulting equation.

Neglecting these terms in equation (13) and going back to the Schrödinger picture, we then get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = {} & -\mathrm{i}[H_{\mathrm{JC}}, \rho(t)] + \sum_{\omega,\omega'\geqslant 0} \{\Gamma(\omega)[A(\omega)\rho(t)A^\dagger(\omega') - A^\dagger(\omega')A(\omega)\rho(t)] + \mathrm{h.c.}\} \\ & + \sum_{\omega,\omega'>0} \{\Gamma(-\omega)[A^\dagger(\omega)\rho(t)A(\omega') - A(\omega')A^\dagger(\omega)\rho(t)] + \mathrm{h.c.}\}.\end{aligned} \tag{18}$$

Inserting the explicit form of the jump operators for our model, we finally obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = {} & -\mathrm{i}[H_{\mathrm{JC}}, \rho(t)] + \sum_{km=\pm 1} \Gamma_{0m}\big(A_{0m}\rho(t)A_{0k}^\dagger - A_{0k}^\dagger A_{0m}\rho(t)\big) \\ & + \sum_{N'\geqslant 1;k,l,m=\pm 1} \Gamma_{0m}\big(A_{0m}\rho(t)A_{N'kl}^\dagger - A_{N'kl}^\dagger A_{0m}\rho(t)\big) \\ & + \sum_{N\geqslant 1;k,m,n=\pm 1} \Gamma_{Nmn}\big(A_{Nmn}\rho(t)A_{0k}^\dagger - A_{0k}^\dagger A_{Nmn}\rho(t)\big) \\ & + \sum_{N,N'\geqslant 1;k,l,m,n=\pm 1} \Gamma_{Nmn}\big(A_{Nmn}\rho(t)A_{N'kl}^\dagger - A_{N'kl}^\dagger A_{Nmn}\rho(t)\big) \\ & + \sum_{k,m=\pm 1} \tilde{\Gamma}_{0m}\big(A_{0m}^\dagger\rho(t)A_{0k} - A_{0k}A_{0m}^\dagger\rho(t)\big) \\ & + \sum_{N'\geqslant 1;k,l,m=\pm 1} \tilde{\Gamma}_{0m}\big(A_{0m}^\dagger\rho(t)A_{N'kl} - A_{N'kl}A_{0m}^\dagger\rho(t)\big) \\ & + \sum_{N\geqslant 1;k,m,n=\pm 1} \tilde{\Gamma}_{Nmn}\big(A_{Nmn}^\dagger\rho(t)A_{0k} - A_{0k}A_{Nmn}^\dagger\rho(t)\big) \\ & + \sum_{N,N'\geqslant 1;k,l,m,n=\pm 1} \tilde{\Gamma}_{Nmn}\big(A_{Nmn}^\dagger\rho(t)A_{N'kl} - A_{N'kl}A_{Nmn}^\dagger\rho(t)\big) + \mathrm{h.c.},\end{aligned} \tag{19}$$

where $\Gamma_{0m} = \Gamma(E_{1,m} - E_0)$, $\tilde{\Gamma}_{0m} = \Gamma(E_0 - E_{1,m})$, $\Gamma_{Nlm} = \Gamma(E_{N+1,l} - E_{N,m})$ and $\tilde{\Gamma}_{Nlm} = \Gamma(E_{N,m} - E_{N+1,l})$ are complex coefficients depending on the thermal spectral density of the environment with which the atom–cavity system interacts, according to

$$\Gamma(\omega) = \pi J(\omega) + \mathrm{i}\mathcal{P}\int_{-\infty}^{+\infty} \mathrm{d}\omega' \frac{J(\omega')}{\omega - \omega'}, \tag{20}$$

where the thermal spectral density $J(\omega) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\omega\tau}\langle E(\tau)E(0)\rangle$ is the Fourier transform of the environment correlation function. This quantity is in general equal to[4]

$$J(\omega) = \begin{cases} (n(\omega)+1)J_0(\omega), & \omega \geqslant 0, \\ n(|\omega|)J_0(|\omega|), & \omega < 0, \end{cases} \tag{21}$$

where $J_0(\omega)$ is the zero-temperature spectral density, defined for nonnegative frequencies, depending only on the density of modes in the reservoir and on the distribution of the system-reservoir coupling constants (i.e., the g_k coefficients in the model given in equation (8)), and $n(\omega) = 1/[\exp(\hbar\omega/k_BT) - 1]$ is the average number of photons in a mode of frequency ω, for a general temperature T. The real part of $\Gamma(\omega)$ is equal to half the decay rate relative to the Bohr frequency ω, while the imaginary part is equal to the corresponding Lamb shift [8].

Equation (19) is the most general equation one can give in accordance with the Born–Markov approximation and a RWA performed on a timescale of the order of ω_0^{-1}. Anyway, equation (19) is not in Lindblad form and in general it might violate the complete positivity requirement for a master equation [8]: the possibility of casting the quasi-RWA master equation in a form suitable for physical applications, i.e. Lindblad form, strongly depends on the specific form of the spectral density of the environment and must be evaluated in specific cases. We will see an important example in the rest of the paper.

4. Comparison: conditions for the validity of the phenomenological model

In order to compare the phenomenological model to the quasi-RWA microscopic model given by equation (19), it is convenient to write equation (4) in the dressed states basis. In the atom–cavity strong coupling regime, this procedure, combined with a secular approximation, leads to the so-called *dressed state approximation* [3, 10, 11], which in [7] we addressed as the main reason why the phenomenological model is so successful in describing the experimental situations.

To this aim let us write the cavity mode annihilation operator with respect to the dressed states basis:

$$a = \sum_{N=0}^{\infty} \sqrt{N+1}|N\rangle\langle N+1| \otimes \mathbb{I}_{\mathrm{at}} = \sum_{m=\pm1} A_{0m} + \sum_{N=1}^{\infty}\sum_{l,m=\pm1} A_{Nlm}, \tag{22}$$

obtained by means of equations (2) and (3) and of the relation $\mathbb{I}_{\mathrm{at}} = (|g\rangle\langle g| + |e\rangle\langle e|)$.

Substituting equation (22) and its Hermitian conjugate into equation (4), one finally obtains the phenomenological master equation in the dressed states basis:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\mathrm{i}[H_{\mathrm{JC}}, \rho(t)] + \gamma(n(\omega_0)+1)\Bigg[\sum_{km=\pm1}\left(A_{0m}\rho(t)A_{0k}^{\dagger} - \frac{1}{2}\left\{A_{0k}^{\dagger}A_{0m}, \rho(t)\right\}\right)$$
$$+ \sum_{N'\geqslant 1;k,l,m=\pm1}\left(A_{0m}\rho(t)A_{N'kl}^{\dagger} - \frac{1}{2}\left\{A_{N'kl}^{\dagger}A_{0m}, \rho(t)\right\}\right)$$

[4] Equations (20) and (21) can be obtained by generalizing the calculations given for the two-level atom master equation in [8, 16].

$$+\sum_{N\geqslant 1;k,m,n=\pm 1}\left(A_{Nmn}\rho(t)A_{0k}^{\dagger}-\frac{1}{2}\{A_{0k}^{\dagger}A_{Nmn},\rho(t)\}\right)$$
$$+\sum_{N,N'\geqslant 1;k,l,m,n=\pm 1}\left(A_{Nmn}\rho(t)A_{N'kl}^{\dagger}-\frac{1}{2}\{A_{N'kl}^{\dagger}A_{Nmn},\rho(t)\}\right)\Bigg]$$
$$+\gamma n(\omega_0)\Bigg[\sum_{k,m=\pm 1}\left(A_{0m}^{\dagger}\rho(t)A_{0k}-\frac{1}{2}\{A_{0k}A_{0m}^{\dagger},\rho(t)\}\right)$$
$$+\sum_{N'\geqslant 1;k,l,m=\pm 1}\left(A_{0m}^{\dagger}\rho(t)A_{N'kl}-\frac{1}{2}\{A_{N'kl}A_{0m}^{\dagger},\rho(t)\}\right)$$
$$+\sum_{N\geqslant 1;k,m,n=\pm 1}\left(A_{Nmn}^{\dagger}\rho(t)A_{0k}-\frac{1}{2}\{A_{0k}A_{Nmn}^{\dagger},\rho(t)\}\right)$$
$$+\sum_{N,N'\geqslant 1;k,l,m,n=\pm 1}\left(A_{Nmn}^{\dagger}\rho(t)A_{N'kl}-\frac{1}{2}\{A_{N'kl}A_{Nmn}^{\dagger},\rho(t)\}\right)\Bigg]. \tag{23}$$

The next step will be a comparison between the phenomenological model in the form of equation (23) and some suitable forms of the microscopic model in equation (19).

From a first comparison between equations (19) and (23), it is very easy to see that the two models coincide when the temperature of the reservoir is zero and its spectrum is flat, a condition strictly related to the Markov approximation, and under the hypothesis that all the imaginary parts of the Γ coefficients, i.e. all the Lamb shifts, are negligible. Indeed under these assumptions all the Γ coefficients in equation (19) are equal to the zero-temperature spectral density J_0 (see equation (21)) and so is half the rate of cavity losses $\gamma/2$ in equation (23). Calling $\gamma/2$ the common value of the Γ coefficients, equation (19) can be rearranged in order to be exactly identical to equation (23) with $n(\omega_0)=0$.

This result provides a condition for the validity of the phenomenological model at zero temperature: it is enough that the spectral density is flat over the Bohr frequencies involved and that all the renormalization terms arising from the imaginary parts of the Γ coefficients, see equation (20), are negligible. The latter condition on the imaginary parts is very often used in the literature, since it is assumed that the Cauchy principal part of an integral of the spectral density is very small compared to the real parts of the coefficients [16]. The assumption of a flat spectrum, instead, depends on the environment with which the atom–cavity system is interacting, and on the initial state of the system: indeed, since in view of the condition $T=0$ the energy of the system cannot increase in time, one is allowed to truncate the state space to a subspace with a finite number of excitations equal to the maximum number of excitations in the initial state, and it is in this space that all the decay rates must be the same.

Let us examine equation (19) from another point of view: the equation is written as a sum of many terms, most of which are counter-rotating. Essentially the counter-rotating terms are all the terms in the sums wherein the downward jump operator is different from the Hermitian conjugate of the upward jump operator in the same term (for example the term $A_{0+}\rho(t)A_{0-}^{\dagger}$ in the first row). What we have previously shown with our analysis is that, under the assumption of zero temperature and flat spectrum, equation (19) can be summed up and can be put in the form of equation (5).

Therefore an aspect emerging from our analysis is that the zero-temperature phenomenological model in equation (5) arises microscopically from an interplay of many counter-rotating terms. This point gives an interesting way of looking at the shift in the frequency of the Rabi oscillations in the atom–cavity system predicted by equation (5) [7].

Indeed in the phenomenological model, because of cavity losses, the Rabi frequency is not exactly equal to 2Ω, but it is equal to $2\Omega\sqrt{1-(\gamma/4\Omega)^2}$, i.e., the Rabi frequency is shifted by a quantity which is second order in the ratio between the decay rate γ and the Rabi splitting Ω. The shift may be interpreted as a cooperative effect of the counter-rotating terms in the dissipator, analogous to the Bloch–Siegert shift appearing in the JC model when counter-rotating terms are added to the atom–cavity interaction Hamiltonian [17].

When the temperature of the reservoir is different from zero, the comparison becomes more difficult to carry on. Let us assume that the imaginary parts of the Γ coefficients are all negligible and that the zero-temperature part $J_0(\omega)$ of the spectral density of the environment is flat. According to equations (20) and (21), the real parts of all the $\Gamma(\omega)$ factors, which are proportional to $J(\omega)$, contain the quantities $n(\omega)+1$ for $\omega \geqslant 0$ and $n(\omega)$ for $\omega < 0$. Therefore in principle one should claim (1) that neither can the Γ coefficients be all equal if $T \neq 0$, nor can the $\tilde{\Gamma}$ coefficients, and (2) that the phenomenological model in equation (4) can never be microscopically justified.

Moreover in this case the microscopic quasi-RWA master equation may not be cast in the Lindblad form. This leads to some problems in its use. On the one hand it may be very difficult to solve, since in general there are not any standard techniques developed for finding solutions of non-Lindblad master equations. On the other hand there is the more fundamental problem that the general form of equation (19) may lead to unphysical results, since it may violate complete positivity [8]. Hence, what we want to look for is a condition under which the phenomenological model is at least approximately valid, so that the problem of the cavity losses can be treated by means of equation (4).

If during its time evolution the atom–cavity system has at most N excitations, the largest Bohr frequency involved in the dynamics is $\omega_0 + (\sqrt{N+1}+\sqrt{N})\Omega$ while the smallest one is $\omega_0 - (\sqrt{N+1}+\sqrt{N})\Omega$. Therefore we have to check the magnitude of the difference between the rates corresponding to the lowest and to the largest Bohr frequency, respectively. Since we are assuming that the quantity $J_0(\omega)$ in equation (21) is flat, the difference in the rates is all due to the difference in the photon populations $n(\omega)$.

Figure 1 shows a plot of the difference in the populations $n(\omega) = 1/[\exp(\hbar\omega/k_B T) - 1]$ of two modes of the reservoir with frequencies ω and $\omega + \delta\omega$ respectively, as a function of the temperature T and of the frequency difference $\delta\omega$. From the plot one can argue that, for low temperatures and not too large values of $\delta\omega$, the error one makes assuming that all the decay rates in the master equation are equal (and assuming the same for the excitation rates) can be considered negligible. Since the $\delta\omega$'s in equation (19) are proportional to $\sqrt{N}$, the negligibility of the difference in the rates imposes that the number of excitations in the atom–cavity system cannot be too large. Therefore we can conclude that, if the difference in the rates can be neglected, the phenomenological model in equation (4) can be considered as a valid model, provided the temperature and the number of total excitations are not too large.

5. Discussion and conclusive remarks

The analysis we have presented shows that, though the microscopic and phenomenological models coincide when the spectrum of the environment is flat, these two models are deeply different. The discrepancy, which involves the predictions when the spectrum is not flat and/or at $T \neq 0$, is conceptual anyway, and relies on the fact that the relevant physical mechanisms are different.

The point of view of our approach is the following: the general formalism of master equations claims that all the decay and decoherence channels involve the atom–cavity system

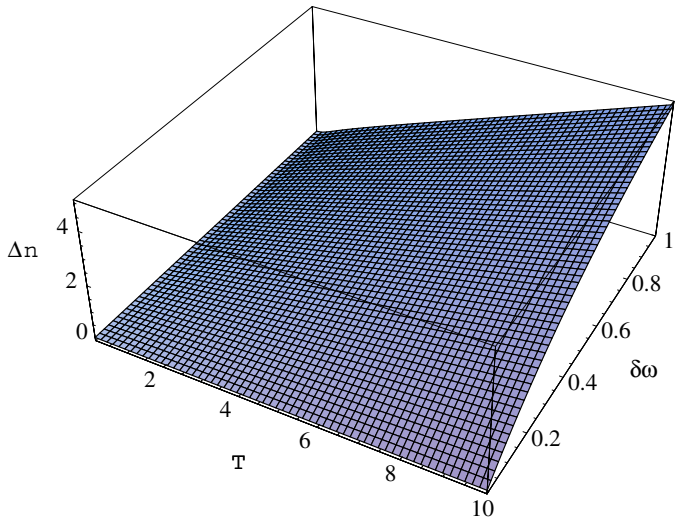

Figure 1. Difference Δn in the average number of photons between a mode of frequency ω and a mode of frequency $\omega + \delta\omega$. The frequency ω is held fixed and the difference Δn is shown as a function of the temperature T (in units of $k_B/\hbar\omega$) and of the frequency difference $\delta\omega$ (in units of ω).

as a whole, and even in the cases wherein it is the cavity only which directly decays also for the microscopic model, this may be seen as a cooperative effect of many non-resonant terms describing the coupling between populations and coherences relative to atom–cavity dressed states.

A last point is worth mentioning. From our analysis we have understood that the success of the phenomenological model relies on the fact that the quasi-RWA microscopic model reduces to it when the spectrum of the environment is flat. The situation is quite different when the spectrum is substantially non-flat. This makes us interested in the possibility of studying the microscopic model in the latter case, for which a non-Markovian theory may be necessary. This is the scope of our future work.

Acknowledgments

M S acknowledges financial support from CIMO for his stay in Turku during the period January–April 2007 and wishes to thank all the Quantum Optics Group members of the University of Turku for their kind hospitality. S M, J P and K A S acknowledge financial support from the Academy of Finland (projects 108699, 115982, 115682), the Magnus Ehrnrooth Foundation and the Väisälä Foundation.

References

[1] Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
[2] For a review see Shore B W and Knight P L 1993 *J. Mod. Opt.* **40** 1195
[3] Haroche S *et al* 1992 *Fundamental Systems in Quantum Optics* ed J Dalibard, J-M Raimond and J Zinn-Justin (Amsterdam: North-Holland)
[4] Raimond J M *et al* 2001 *Rev. Mod. Phys.* **73** 565
[5] Walther H *et al* 2006 *Rep. Prog. Phys.* **69** 1325

[6] Wineland D *et al* 1998 *J. Res. Natl. Inst. Stand. Technol.* **103** 258
[7] Scala M *et al* 2007 *Phys. Rev.* A **75** 013811
[8] Breuer H-P and Petruccione F 2002 *The Theory of Open Quantum Systems* (Oxford: Oxford University Press)
[9] Cohen-Tannoudji C *et al* 1998 *Atom-Photon Interactions* (New York: Wiley)
[10] Puri R R and Agarwal G S 1986 *Phys. Rev.* A **33** 3610
[11] Puri R R and Agarwal G S 1987 *Phys. Rev.* A **35** 3433
[12] Agarwal G S and Puri R R 1986 *Phys. Rev.* A **33** 1757
[13] Barnett S M and Knight P L 1986 *Phys. Rev.* A **33** 2444
[14] Briegel H-J and Englert B-G 1993 *Phys. Rev.* A **47** 3311
[15] Puri R R 2001 *Mathematical Methods of Quantum Optics* (Berlin: Springer)
[16] Gardiner C W and Zoller P 2000 *Quantum Noise* (Berlin: Springer)
[17] Allen L and Eberly J H 1975 *Optical Resonance and Two-Level Atoms* (New York: Wiley)